

The University of Chicago

PARTIALLY DEGENERATE STELLAR MODELS

A PART OF A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE PHYSICAL
SCIENCES IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF ASTRONOMY AND ASTROPHYSICS
DECEMBER, 1940

By
GORDON W. WARES

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ABSTRACT

Partially degenerate stellar models based on the Fermi-Dirac equation of state of an electron gas are used to study stars that are too degenerate at the center for the perfect gas law to apply but not dense enough for the white-dwarf models to be valid. The relation of the Fermi-Dirac equation of state to the other equations of state of an electron gas, together with the percentage of error committed in using them, is presented in Figure 1. The $\log \rho, \log T$ diagram of Figure 1 illustrates and summarizes chapter x on the quantum statistics in Chandrasekhar's monograph on stellar structure, which is basic to the present investigation.

In Part I the theory of the standard-model case is developed. Table 1 and Figure 2 present the three numerical integrations, and Table 2 gives the boundary values. Section 11, Table 3, and Figures 3 and 4 present the accurate solution of Milne's problem (for small masses), namely, the determination of the course of the $(\mathfrak{R}, 1 - \beta)$ curves of constant mass in the domain of degeneracy. This constitutes our most important result.

In Part II a parallel development of the isothermal case is given. The results of seven numerical integrations are summarized in Figure 5 and Table 4. All configurations are found to extend to infinity mathematically, but finite boundaries can be defined for not too low central degeneracy.

In Part III it is shown that the present theory is required for the more extreme subdwarfs and perhaps for the old novae, but probably not for any main-sequence stars.

1. *Partially degenerate stellar models.*—It is the purpose of the present paper to use partially degenerate stellar models based on the Fermi-Dirac equation of state of an electron gas to study stars that are too degenerate at the center for the perfect gas law

$$P = \frac{k}{\mu_e H} \rho T \quad (1)$$

to apply but not dense enough for the white-dwarf models, based on the equation of state²

$$\rho = Bx^3, \quad p = Af(x), \quad (2)$$

of a completely degenerate electron gas to be valid; in particular it is our purpose to solve Milne's problem (cf. § 11).

In Figure 1 and its caption are summarized and illustrated (a) the various relatively simple forms to which the general equation of state of an electron gas reduces in each of the "domains" of the $\log \rho, \log T$ diagram, (b) the "criteria" defining the boundaries between domains, and (c) the series expansions required to "approach" and the integral formulae required to actually "cross" the degeneracy-criterion locus. Figure 1 is useful not only in telling at a glance the correct equation of state for any point ($\log \rho, \log T$) but also in giving, by inspection, the percentage of error committed in using either the perfect gas law or the completely degenerate gas law.

It is anticipated that for the above-mentioned stars of intermediate density the central density and temperature will correspond to a point ($\log \rho_c, \log T_c$) dangerously close to, if not actually below, the degeneracy-criterion locus of Figure 1. It is also anticipated

¹ The present paper is a condensation of the writer's doctoral dissertation, "Partially Degenerate Stellar Models" (University of Chicago, 1940). It was completed while the author was in military service.

² In general, the notation in this paper is that of S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (Chicago: University of Chicago Press, 1939) (hereafter "*Stellar Structure*").

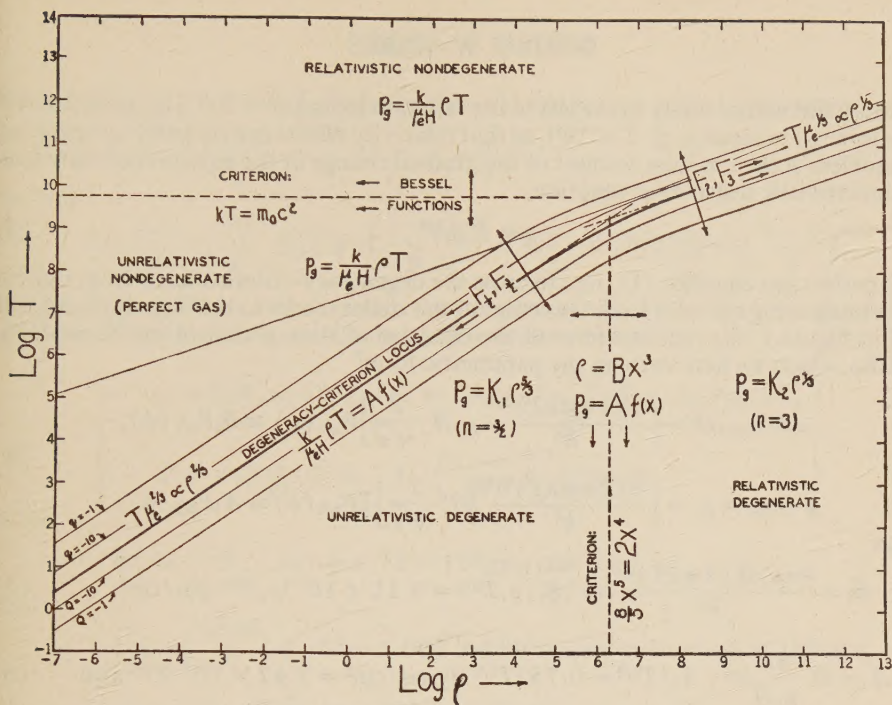


FIG. 1.—The $\log \rho$, $\log T$ diagram of the equations of state of an electron gas. The diagram is based upon and illustrates chapter x of *Stellar Structure*, to which the equation numbers below refer. The stellar degeneracy-criterion locus, p_0 (perfect gas) $= k/(\mu_e H) \rho T = Af(x) = p_0$ (degenerate), divides the diagram into two domains: nondegenerate (above) and degenerate (below). These two domains correspond to two limiting cases in which it is possible to integrate in series the general equation of state (170) and (172), which is valid over the entire diagram. Over the entire nondegenerate domain the Bessel function infinite series form (252) and (253) is valid; and over the entire degenerate domain the algebraic infinite series form, $\rho = Bx^3$ [1 + ...], $p_0 = Af(x)$ [1 + ...], (204) and (198), is valid. The equation of state is known in integrated form, therefore, for every point on the diagram, but each series breaks down if one attempts to “cross” the degeneracy-criterion locus. To effect this crossing, i.e., to take exact account of partial degeneracy, a numerical quadrature of (170) and (172) is always required. In the two limiting cases of low density and temperature and of high density and temperature this quadrature involves but a single parameter and is expressed in terms of the Fermi-Dirac functions $F_{1/2}$, $F_{3/2}$, and F_2 , F_3 , respectively, which have been tabulated in the former case. The partially degenerate stellar models of the present paper are based upon the $F_{1/2}$ and $F_{3/2}$ functions. Sufficiently far ($\psi \gg 1$) below the degeneracy-criterion locus the algebraic infinite series reduce approximately to their first terms, namely, the relativistic completely degenerate gas law (white-dwarf equation of state), $p_0 = Bx^3$, $p_0 = Af(x)$, which always gives the pressure too low, the error being $Q\% = -1\%$ and $q\% = -10\%$ along the loci labeled $Q = -1$ and $Q = -10$ on the diagram. The two criterion-loci, $(8/5)x^5 = 2x^4$ at $\log \rho = 6.2828$ ($\rho \approx 2 \times 10^6$) and $kT = m_0 c^2$ at $\log T = 9.773$ ($T \approx 6 \times 10^8$), indicate the setting-in of the relativity correction to mass on account of high velocity caused by high density (Pauli exclusion principle) and high temperature, respectively. To the left ($\rho \ll 2 \times 10^6$) and to the right ($\rho \gg 2 \times 10^6$) of the criterion-locus $(8/5)x^5 = 2x^4$ the equation of state $\rho = Bx^3$, $p_0 = Af(x)$ reduces to $p_0 = K_1 \rho^{5/3}$ (unrelativistic degenerate or polytrope $n = 3/2$) and to $p_0 = K_2 \rho^3$ (relativistic degenerate or polytrope $n = 3$), respectively. Sufficiently far ($\psi \ll -1$) above the degeneracy-criterion locus the Bessel function infinite series reduce approximately to their first terms, namely, the perfect gas law, $p_0 = k/(\mu_e H) \rho T$, which always gives the pressure too low, the error being $q\% = -1\%$ and $q\% = -10\%$ along the loci labeled $q = -1$ and $q = -10$ on the diagram. In general, the physical properties of an electron gas have to be expressed in terms of Bessel functions in the presence of relativity effects due to high temperature, i.e., above and in the vicinity of the criterion-locus $kT = m_0 c^2$. The perfect gas equation of state, however, is unaltered by such relativity effects. Hence the $kT = m_0 c^2$ locus would be of little importance for stellar structure, even if temperatures of 10^9 were of interest. The locus of $s\% = -1\%$ error caused by relativity effects due to high density in the pressure calculated from the Fermi-Dirac $F_{3/2}$ function passes through the points (5.74, 8) and (5.27, 6) but is not indicated on the diagram. The error is -10 per cent at the point (5.84, 6). The light line passing through the perfect gas domain with slope $\frac{1}{3}$ is the locus of minimum radius for given ψ_0 (for $\mu_e = 1$) for the partially degenerate standard model (cf. § 9). The $q = -1$, $q = -10$, $k/(\mu_e H) \rho T = Af(x)$, $Q = -10$, and $Q = -1$ loci correspond, respectively, to the following constant values of the parameter ψ in the region $\rho \ll 2 \times 10^6$: -2.852 , -0.371 , $+2.099$, $+6.28$, and $+20.25$.

that this point will be safely to the left of the criterion locus $\frac{8}{3}x^5 = 2x^4$ (i.e., $\rho_c \ll 2 \times 10^6$, which is found to mean $\rho_c \leq 2 \times 10^5$), so that relativity effects can properly be neglected. The problem is then to take account of the gradual change in the equation of state from the unrelativistic degenerate equation,

$$p = K_1 \rho^{5/3}, \quad (3)$$

to the perfect gas equation (1) (i.e., to cross the degeneracy-criterion locus from right to left) in integrating outward from the center of the stellar model to the boundary. As indicated in Figure 1, the required form of the equation of state is that of the Fermi-Dirac statistics, which we here write in the parametric form³

$$\left. \begin{aligned} \rho &\equiv n_e \mu_e H = \left[\frac{2 (2\pi m k T)^{3/2}}{h^3} \mu_e H \frac{2}{\sqrt{\pi}} \right] F_{1/2}(\psi) \equiv B_1 F_{1/2}(\psi), \\ p &= p_o = p_e = \left[\frac{2 (2\pi m k T)^{3/2}}{h^3} k T \frac{2}{\sqrt{\pi}} \right] \frac{2}{3} F_{3/2}(\psi) \equiv A_1 F_{3/2}(\psi), \end{aligned} \right\} \quad (4)$$

where⁴

$$B_1 = \frac{4\pi \mu_e H (2 m k T)^{3/2}}{h^3} \equiv |B_1| \mu_e T^{3/2} = 9.11 \times 10^{-9} \mu_e T^{3/2} \text{ gm/cm}^3 \quad (5)$$

and

$$A_1 = B_1 \frac{kT}{\mu_e H} \equiv |A_1| T^{5/2} = 0.752 T^{5/2} \text{ dynes/cm}^2 = 7.42 \times 10^{-7} T^{5/2} \text{ atm} \quad (6)$$

and where $F_{1/2}(\psi)$ and $\mathfrak{F}_{3/2}(\psi) \equiv \frac{2}{3} F_{3/2}(\psi)$ are the Fermi-Dirac functions defined by

$$F_v(\psi) = \int_0^\infty \frac{u^v du}{e^{-\psi+u} + 1}. \quad (7)^5$$

In order to calculate the structure of configurations obeying the parametric equation of state (4) (the equation for pressure being modified by the addition of the radiation pressure $(a/3)T^4$), we must make an assumption as to the temperature gradient and the distribution of energy sources; to each such assumption there will correspond a particular "partially degenerate stellar model." In this paper we shall study two such models: (I) the standard model and (II) the isothermal gas sphere. The former leads to the most important specific result of the present paper: the solution of Milne's problem (cf. § 11).

I. THE PARTIALLY DEGENERATE STANDARD MODEL

2. *The fundamental differential equation.*—The definitive condition of the standard model may be taken as

$$p_o = \beta P, \quad \beta = \text{constant}, \quad (8)$$

where $P = p_o + p_r$ is the total pressure. The equation of state of the model, corresponding to the equation of state (4) of the electron gas, is

$$\rho = |B_1| \mu_e T^{3/2} F_{1/2}(\psi), \quad P = |A_1| \beta^{-1} T^{5/2} \mathfrak{F}_{3/2}(\psi). \quad (9)$$

³ These equations are equivalent to eqs. (151) and (152), p. 447, of *Stellar Structure*.

⁴ In the present paper, symbols like $A_1, B_1, C_1, \alpha_1, A_2, B_2, C_2$, and α_2 represent quantities that for a particular model (subscripts 1 for isothermal gas sphere and 2 for standard model) are constants involving parameters. The corresponding symbols $|A_1|, |B_1|$, etc., represent the numerical values of these constants apart from parameters but including physical constants. The physical constants (in cgs units) adopted in this paper are the same as those used in Appen. I of *Stellar Structure*.

⁵ The functions $\frac{2}{3} F_{3/2}(\psi)$, abbreviated in this paper to $\mathfrak{F}_{3/2}(\psi)$, or $\mathfrak{F}_{3/2}$, or $\mathfrak{F}$; and $F_{1/2}(\psi)$, abbreviated to $F_{1/2}$, or F , are tabulated by J. McDougall and E. C. Stoner in *Phil. Trans.*, A, 257, 67, 1938.

Using for the gas pressure p_g from equation (4) and for the radiation pressure

$$p_r = \frac{a}{3} T^4, \quad (10)$$

we find

$$T = \left(\frac{3}{a} |A_1| \right)^{2/3} \mathfrak{F}_{3/2}^{2/3}(\psi) \left(\frac{1-\beta}{\beta} \right)^{2/3} = 4.47 \times 10^9 \mathfrak{F}_{3/2}^{2/3}(\psi) \left(\frac{1-\beta}{\beta} \right)^{2/3} \quad (11)$$

by means of which we can eliminate the temperature from (9). We thus obtain for the equation of state of the partially degenerate standard model:

$$\left. \begin{aligned} \rho &= B_2 \mathfrak{F}_{3/2}(\psi) F_{1/2}(\psi) \equiv |B_2| \mu_e \frac{1-\beta}{\beta} \mathfrak{F}_{3/2}(\psi) F_{1/2}(\psi), \\ P &= A_2 \mathfrak{F}_{3/2}^{8/3}(\psi) \equiv |A_2| \left(\frac{1-\beta}{\beta^{8/5}} \right)^{5/3} \mathfrak{F}_{3/2}^{8/3}(\psi), \end{aligned} \right\} \quad (12)$$

where

$$|B_2| = \frac{3}{a} |B_1| |A_1| = 2.72 \times 10^6 \text{ gm/cm}^3 \quad (13)$$

and

$$|A_2| = \left(\frac{3}{a} \right)^{5/3} |A_1|^{8/3} = 1.003 \times 10^{24} \text{ dynes/cm}^2 = 9.89 \times 10^{17} \text{ atm.} \quad (14)$$

Using equation (12) and the recurrence relations⁶

$$F'_v(\psi) = v F_{v-1}(\psi), \quad \mathfrak{F}'_{3/2}(\psi) = F_{1/2}(\psi) \quad \text{or} \quad \mathfrak{F}' = F, \quad (15)$$

and proceeding in the usual manner, introducing the dimensionless variable ξ through the relation

$$r = a_2 \xi, \quad (16)$$

where

$$a_2 = \left[\frac{2 A_2}{3 \pi G B_2^2} \right]^{1/2} = \frac{|a_2|}{\mu_e [\beta^2 (1-\beta)]^{1/6}} = \frac{0.00944 R_\odot}{\mu_e [\beta^2 (1-\beta)]^{1/6}}, \quad (17)$$

we obtain for the two equivalent forms of the fundamental differential equation of the partially degenerate standard model:

$$\left. \begin{aligned} \frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \mathfrak{F}_{3/2}^{2/3}(\psi) \frac{d\psi}{d\xi} \right) &= - \mathfrak{F}_{3/2}(\psi) F_{1/2}(\psi); \\ \frac{d^2\psi}{d\xi^2} + \frac{2}{3} \frac{F_{1/2}(\psi)}{\mathfrak{F}_{3/2}(\psi)} \left(\frac{d\psi}{d\xi} \right)^2 + \frac{2}{\xi} \frac{d\psi}{d\xi} &= - \mathfrak{F}_{3/2}^{1/3}(\psi) F_{1/2}(\psi). \end{aligned} \right\} \quad (18)$$

The required solution⁷ is the function $\psi(\xi)$ with the usual boundary conditions:

$$\left. \begin{aligned} \text{at the center } (\xi = 0): \quad \psi(0) &= \psi_0, \quad \left(\frac{d\psi}{d\xi} \right)_0 = 0; \\ \text{at the boundary } (\xi = \xi_1): \quad \psi(\xi_1) &= -\infty. \end{aligned} \right\} \quad (19)$$

There is no homology relation.

⁶ McDougall and Stoner, *ibid.*, eq. (6.3).

⁷ We shall concern ourselves only with solutions analogous to the *E*-solutions of polytropic theory; cf. *Stellar Structure*, pp. 103 and 159.

For given values of the relative radiation pressure $(1 - \beta)/\beta$ and molecular weight μ_e it is clear from equations (11) and (12) that the central values ρ_c , P_c , and T_c depend upon the choice of the integration parameter ψ_0 , which is a measure of the degree of degeneracy at the center. Conversely, we can think of a given ψ_0 as follows:

$$\psi_0 \text{ specifies } \frac{\rho_c}{\mu_e \frac{1 - \beta}{\beta}} \text{ and } \frac{P_c}{\left(\frac{1 - \beta}{\beta^{8/5}}\right)^{5/3}} \text{ and } \frac{T_c}{\left(\frac{1 - \beta}{\beta}\right)^{2/3}}, \text{ or } \frac{\rho_c}{\mu_e T_c^{3/2}}, \quad (20)$$

the second and third quantities being independent of μ_e and the last independent of $(1 - \beta)$.

3. *The density at any point in terms of the central density.*—By equation (19), the first of equations (12), and the fact that β and μ_e are assumed constant throughout the configuration, we find that

$$\frac{\rho}{\rho_c} = \frac{\mathfrak{F}_{3/2}(\psi) F_{1/2}(\psi)}{\mathfrak{F}_{3/2}(\psi_0) F_{1/2}(\psi_0)}. \quad (21)$$

4. *The mass relation or "generalized quartic equation" for $(1 - \beta)$.*—By equation (16) the mass inclosed in a sphere of radius ξ is given by

$$M(\xi) = 4\pi a_2^3 \int_0^\xi \rho \xi^2 d\xi, \quad (22)$$

or, according to equations (12) and (18), by

$$M(\xi) = -C_2 \mathfrak{F}_{3/2}^{2/3}(\psi) \xi^2 \frac{d\psi}{d\xi} = -|C_2| \mu_e^{-2} \left(\frac{1 - \beta}{\beta^4}\right)^{1/2} \mathfrak{F}_{3/2}^{2/3}(\psi) \xi^2 \frac{d\psi}{d\xi}, \quad (23)$$

where

$$|C_2| = 4\pi |a_2|^3 |B_2| = 9.67 \times 10^{33} \text{ gm} = 4.87 \odot. \quad (24)$$

The mass of the entire configuration is given by

$$M = -|C_2| \mu_e^{-2} \left(\frac{1 - \beta}{\beta^4}\right)^{1/2} [\mathfrak{F}_{3/2}^{2/3}(\psi_1) \xi_1^2 \psi'(\xi_1)], \quad (25)$$

which is a quartic equation in β determining the relative radiation pressure $(1 - \beta)$ uniquely in terms of the mass $M\mu_e^2$, or vice versa, for a given solution ψ_0 . Equation (25) may be considered as a generalization of Eddington's quartic equation for the classical standard model. It is, however, important to note that M is not independent of the radius and central density, as is the case with the classical standard model.

5. *The relation between the mean and the central density.*—From elementary considerations involving equations (12), (16), and (25) we find

$$\frac{\rho_c}{\rho} = -\mathfrak{F}_{3/2}(\psi_0) F_{1/2}(\psi_0) \left[\frac{\xi_1}{3\psi'(\xi_1) \mathfrak{F}_{3/2}^{2/3}} \right]. \quad (26)$$

6. *The limiting cases of very low and of very high central degeneracy.*—It is not difficult to verify that the partially degenerate standard model reduces as it should to the classical standard model or Lane-Emden polytrope $n = 3$ as $\psi \rightarrow -\infty$ and in the opposite limiting case of $\psi \rightarrow +\infty$ and for negligible radiation pressure ($\beta \rightarrow 1$) to a Lane-Emden polytrope $n = \frac{3}{2}$ or white-dwarf configuration in the limiting case of low central density.

Thus, our partially degenerate standard-model configurations based on the Fermi-Dirac equation of state (12) and described by $\psi(\xi)$ of equation (18) are appropriate for

taking into exact account the effects of incomplete degeneracy. The white-dwarf configurations based on equation (2) take complete account of relativity effects but break down at "low density" (strictly low degeneracy) with approach from the right (Fig. 1) to the degeneracy-criterion locus. In a somewhat complementary manner our partially degenerate configurations based on equation (4) take exact account of incompleteness of degeneracy but break down at high density with the approach from the left to the $\frac{8}{5}x^5 = 2x^4$ criterion locus on account of the setting-in of relativity effects. In the region of intermediate density, where the two conditions $\psi \gg 1$ and $\rho \ll 2 \times 10^6$ are both satisfied, the white-dwarf and the partially degenerate configurations are equally valid.

7. *The numerical integrations.*—The following starting series for the standard-model differential equation (18) was derived and used:

$$\psi(\xi) = \psi_0 + a\xi^2 + b\xi^4 + c\xi^6 + g\xi^8 + \dots, \quad (27)$$

where (using $\mathfrak{F}' = F$)

$$\left. \begin{aligned} a &= -\frac{1}{3!} \mathfrak{F}^{1/3} F, & b &= +\frac{1}{6!} \frac{2}{3} \mathfrak{F}^{-1/3} [9 \mathfrak{F} F F' - F^3], \\ c &= -\frac{1}{9!} \frac{8}{9} \left[103 \frac{F^5}{\mathfrak{F}} - 111 F^3 F' + 135 \mathfrak{F} F^2 F'' + 81 \mathfrak{F} F (F')^2 \right], \\ g &= -\left\{ \frac{a^4}{54 \mathfrak{F}^3} [2F^3 - 3 \mathfrak{F} F F' + \mathfrak{F}^2 F''] + \frac{a^3}{11,664 \mathfrak{F}^{2/3}} \left[\frac{10F^4}{\mathfrak{F}^2} - \frac{36F^2 F'}{\mathfrak{F}} \right. \right. \\ &\quad \left. \left. + 27 (F')^2 + 36 F F'' + 27 \mathfrak{F} F''' \right] + \frac{5a^2 b}{27 \mathfrak{F}} \left[F' - \frac{F^2}{\mathfrak{F}} \right] + \frac{a b}{648 \mathfrak{F}^{2/3}} [9 F F' \right. \\ &\quad \left. - 2 \frac{F^3}{\mathfrak{F}} + 9 \mathfrak{F} F''] + \frac{2a c F}{9 \mathfrak{F}} + \frac{4 b^2 F}{27 \mathfrak{F}} + \frac{c}{216 \mathfrak{F}^{2/3}} [\mathfrak{F}^2 + 3 \mathfrak{F} F'] \right\}. \end{aligned} \right\} \quad (28)$$

For any ψ_0 , all the required values⁸ of $F = F_{1/2}(\psi_0)$ and its derivatives and of $\mathfrak{F} \equiv \mathfrak{F}_{3/2}(\psi_0)$ can be obtained by interpolation in the tables of McDougall and Stoner.

The numerical integrations have been carried out for the three values 0, 2, and 5 of the parameter ψ_0 . In Table 1 the following measures of physical quantities: ψ , $\mathfrak{F}_{3/2}(\psi) F_{1/2}(\psi)$ (density) and $-\mathfrak{F}_{3/2}^{2/3}(\psi) \xi^2 (d\psi/d\xi)$ (mass inside radius ξ) are tabulated as functions of ξ (radius) at intervals sufficiently close to permit interpolation if differences of high-enough order are used. The relative density-distribution-curves are shown in Figure 2. Table 2 gives the following boundary-value constants: ξ_1 ; $-\mathfrak{F}_{3/2}^{2/3}(\psi_1) \xi_1^2 \psi'(\xi_1)$; the constant ratio M/M_E , of the mass to the Eddington mass; $\rho_c/\bar{\rho}$ (calculated from eq. [26]); and the proportionality constants $J_M(\psi_0)$ between $\mathfrak{M} \equiv M \mu_e^2/\odot$ and $(1-\beta)^{1/2}$ and $J_E(\psi_0)$ between $\mathfrak{R} \equiv R \mu_e/R\odot$ and $(1-\beta)^{-1/6}$, respectively (calculated from eq. [34] below).

8. *The mass-radius relation.*—From equation (25) and from equations (16) and (17) we have the following pair of simultaneous equations determining the mass-radius relation:

$$\frac{M \mu_e^2}{|C_2|} = -[\mathfrak{F}_{3/2}^{2/3}(\psi_1) \xi_1^2 \psi'(\xi_1)] \left(\frac{1-\beta}{\beta^4} \right)^{1/2}; \quad \frac{R \mu_e}{|a_2|} = \frac{\xi_1}{[\beta^2(1-\beta)]^{1/6}}. \quad (29)$$

The limiting cases $\psi_0 \gg 1$ and $\psi_0 \ll -1$ may be noted. In the former case,

$$\frac{M \mu_e^2/|C_2|}{(R \mu_e/|a_2|)^{-3}} = -\frac{243}{2048} [\eta_1^5 \theta'_{3/2}(\eta_1)] \beta^{-3} = 5.2359 \beta^{-3} \quad (\psi_0 \gg 1), \quad (30)$$

⁸ In eq. (28) we have used the abbreviation F for $F_{1/2}(\psi_0)$ and $\mathfrak{F}$ for $\mathfrak{F}_{3/2}(\psi_0)$. Elsewhere in this paper F and $\mathfrak{F}_{3/2}$ refer to the variable ψ .

TABLE 1
THE PARTIALLY DEGENERATE STANDARD-MODEL FUNCTION

ξ	ψ	$\mathfrak{F}_{3/2}F_{1/2}$	$-\mathfrak{F}^{2/3}\xi^2\psi'$	ξ	ψ	$\mathfrak{F}_{3/2}F_{1/2}$	$-\mathfrak{F}^{2/3}\xi^2\psi'$
$\psi_0=0$							
0.0.....	-0.000000	0.521140	0.0000	5.5.....	-2.11297	0.010790	4.0083
0.2.....	0.004137	.517544	0.0014	6.0.....	2.41950	.0059380	4.1420
0.4.....	0.016507	.506932	0.0109	6.5.....	2.74737	.0031210	4.2273
0.6.....	0.036982	.489806	0.0362	7.0.....	3.10524	.00154078	4.2782
0.8.....	0.065359	.466956	0.0834	7.2.....	3.25972	.00113501	4.2916
1.0.....	0.101362	.439390	0.1569	7.4.....	3.42247	.00082211	4.3019
1.2.....	0.144655	.408248	0.2596	7.6.....	3.59516	.00058361	4.3098
1.4.....	0.194849	.374719	0.3919	7.8.....	3.77986	.00040432	4.3156
1.6.....	0.251520	.339960	0.5529	8.0.....	3.97942	.00027185	4.3197
1.8.....	0.314219	.305031	0.7388	8.2.....	4.19766	.00017604	4.3226
2.0.....	0.382484	.270845	0.9464	8.4.....	4.43999	.00010861	4.3246
2.5.....	0.574494	.193021	1.5258	8.6.....	4.71439	.00006283	4.3258
3.0.....	0.791662	.130637	2.1270	8.8.....	5.03342	.00003324	4.3265
3.5.....	1.02833	.084707	2.6836	9.0.....	5.41855	.00001540	4.3269
4.0.....	1.28038	.052991	3.1561	9.2.....	5.91119	.00000576	4.3270
4.5.....	1.54536	.032127	3.5303	9.4.....	6.60944	.00000143	4.3271
5.0.....	-1.82256	0.018909	3.8100	9.6.....	-7.86857	0.00000011	4.3271
$\psi_0=2$							
0.0.....	+2.000000	9.23783	0.00000	1.4.....	+0.904018	2.15574	3.7699
0.1.....	1.993560	9.16688	0.00307	1.5.....	+0.763085	1.74991	4.1786
0.2.....	1.974304	8.95746	0.02419	1.6.....	+0.616582	1.40156	4.5553
0.3.....	1.942429	8.61965	0.07977	1.7.....	+0.464697	1.10739	4.8950
0.4.....	1.898251	8.16925	0.18314	1.8.....	+0.307484	0.86281	5.1948
0.5.....	1.842198	7.62650	0.34346	1.9.....	+0.144827	0.66253	5.4541
0.6.....	1.774789	7.01448	0.56510	2.0.....	-0.023569	0.50096	5.6737
0.7.....	1.696621	6.35743	0.84754	2.2.....	-0.37990	0.272069	6.003
0.8.....	1.608344	5.67912	1.18577	2.4.....	-0.76918	0.136069	6.211
0.9.....	1.510641	5.00143	1.57103	2.6.....	-1.20613	0.060889	6.328
1.0.....	1.404202	4.34332	1.99184	2.8.....	-1.71907	0.023063	6.385
1.1.....	1.289707	3.72016	2.4352	3.0.....	-2.36892	0.0065549	6.407
1.2.....	1.167798	3.14337	2.8877	3.2.....	-3.32592	0.0009955	6.413
1.3.....	+1.039063	2.62055	3.3364	3.4.....	-5.62393	0.0000102	6.414
$\psi_0=5$							
0.0.....	+5.00000	145.2766	0.00000	0.95....	+2.33338	13.5951	12.776
0.1.....	4.96551	141.7900	0.04773	1.00....	+2.07041	10.0441	13.334
0.2.....	4.86303	131.8135	0.36557	1.05....	+1.7915	7.1624	13.782
0.3.....	4.69537	116.6844	1.1484	1.10....	+1.4929	4.88616	14.126
0.4.....	4.46685	98.2924	2.4656	1.15....	+1.1687	3.14716	14.377
0.5.....	4.18271	78.6910	4.2506	1.20....	+0.8094	1.87500	14.547
0.6.....	3.84843	59.7307	6.3290	1.25....	+0.3991	0.99851	14.652
0.7.....	3.46885	42.8066	8.4720	1.30....	-0.0918	0.44653	14.709
0.80....	3.04704	28.7528	10.455	1.35....	-0.7272	0.14681	14.733
0.85....	2.82044	22.9143	11.331	1.40....	-1.6904	0.02436	14.740
0.90....	+2.58286	17.8707	12.107	1.45....	-4.2456	0.00016	14.742

which shows that the mass is inversely proportional to the cube of the radius. The condition $\psi_0 \gg 1$, or its equivalent form: $\rho_c/\mu_e (1 - \beta/\beta) \gg |B_2| \approx 3 \times 10^6$ implies that $1 - \beta$ must be very small in order that $\rho_c \mu_e^{-1}$ and T_c be much smaller than their criterion values. For $\rho_c \mu_e^{-1}$ equals one-tenth of its criterion value of 2×10^6 , $1 - \beta =$

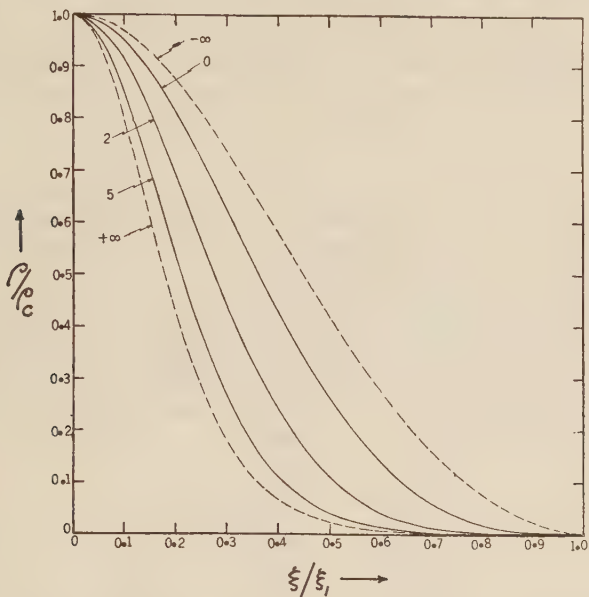


FIG. 2.—Relative density distributions for the partially degenerate standard model for the indicated values of ψ_0 .

TABLE 2

THE BOUNDARY-VALUE CONSTANTS OF THE PARTIALLY DEGENERATE STANDARD MODEL

ψ_0	ξ_1	$-\delta_{3/2}^{2/3}(\psi_1)\xi_1^2\psi'(\xi_1)$	$\frac{M}{M_E}$	$\frac{\rho_c}{\bar{\rho}}$	$J_M(\psi_0)$	$J_R(\psi_0)$
$-\infty$ ($n=3$)	∞	3.707736	1	54.18248	17.98	
0	9.75789	4.3271	1.1670	37.300	21.08	0.0921
2	3.45971	6.414 ₃	1.7300	19.880	31.2	.0327
5	1.4617	14.74 ₂	3.976	10.26	71.8	0.01380
$+\infty$ ($n=\frac{3}{2}$)	0	∞	∞	5.99071		

0.1, 0.008, and 0.0005 for $\psi_0 = 0, 2$, and 5 , respectively; for T_c equals one-tenth of its criterion value of 6×10^9 , $1 - \beta = 0.07, 0.02$, and 0.003 for $\psi_0 = 0, 2$, and 5 , respectively. In any case, for negligible radiation pressure, we readily verify that equation (30) reduces precisely to the well-known mass-radius relation⁹ of a Lane-Emden polytrope with index $n = \frac{3}{2}$ and $K = K_1$, which we have seen is appropriate in the present case. In the opposite limiting case, $\psi_0 \ll -1$, we find

$$\frac{M\mu_e^2/C_2}{(R\mu_e/a_2)^{-3}} = \frac{\frac{27}{2\pi}[-\eta_1^5\theta_3'(\eta_1)]}{\beta^3 e^{2\psi_0}} = 2845.17\beta^{-3}e^{-2\psi_0} \quad (\psi_0 \ll -1), \quad (31)$$

⁹ *Stellar Structure*, eq. (72), p. 98.

which breaks down in the limit $\psi \rightarrow -\infty$, corresponding to the well-known fact that there is no mass-radius relation for the classical standard model.

9. *The $\log \rho_c, \log T_c$ locus of minimum radius for given central degeneracy (ψ_0).*—According to equation (29), the value $(1 - \beta) = \frac{1}{3}$ makes the radius a minimum for a given value of ξ_1 and hence of ψ_0 . Hence, we set $(1 - \beta) = \frac{1}{3}$ in equation (11) and the first of equations (12) and obtain as the locus on the $\log \rho, \log T$ diagram (Fig. 1) of pairs of values of ρ_c and T_c that make the radius of the star a minimum for a given degree of central degeneracy (ψ_0) the straight line

$$\log T_c = \frac{1}{3} \log (\rho_c \mu_e^{-1}) - 7.66 \quad (\rho_c \ll 2 \times 10^6). \quad (32)$$

This line necessarily cuts across the lines of constant ψ_0 , all of which have slope $\frac{2}{3}$ (parallel to the $T \mu_e^{2/3} \propto \rho^{2/3}$ portion of the degeneracy-criterion locus), with a slope just one-half as great when $\mu_e = 1$. The linear portion (eq. [32]), as well as the upward-curving extrapolated portion, of the locus is shown in Figure 1.

10. *The approximate mass-radius relation for $(1 - \beta) \ll 1$.*—For $(1 - \beta) = 0.006$, corresponding at the upper limit to masses of $1.41 \odot$ for $\psi_0 = -\infty$ (Eddington mass) and to $5.63 \odot$ for $\psi_0 = 5$, we can set $\beta = 1$ in equations (29) and study the variations of mass and radius with $(1 - \beta)$ and with each other with a maximum error of 1.2 per cent at $(1 - \beta) = 0.006$. We can write equations (29) for the mass $\mathfrak{M}$ and radius $\mathfrak{R}$, both measured in solar units, in the form

$$\left. \begin{aligned} \mathfrak{M} &\equiv \frac{M \mu_e^2}{\odot} = J_M(\psi_0) (1 - \beta)^{1/2}; \\ \mathfrak{R} &\equiv \frac{R \mu_e}{R_\odot} = J_R(\psi_0) (1 - \beta)^{-1/6} \end{aligned} \right\} \quad (1 - \beta \ll 1), \quad (33)$$

where

$$J_M(\psi_0) = \frac{|C_2|}{\odot} [\mathfrak{F}_{3/2}^{2/3}(\psi_1) \xi_1^2 \psi'(\xi_1)]; \quad J_R(\psi_0) = \frac{|a_2|}{R_\odot} \xi_1 \quad (\psi_0 = \psi_0). \quad (34)$$

Using Table 2, we can write equation (29) in numerical form convenient for the determination of the $(\mathfrak{R}, 1 - \beta)$ curves for given mass. For the values of ψ_0 for which boundary values are known we have

$$\left. \begin{aligned} \psi_0 = \infty: & \left\{ \begin{aligned} 1 - \beta &= 0 \\ \mathfrak{R} &= 0.04009 \mathfrak{M}^{-1/3} \equiv \mathfrak{R}_1 \end{aligned} \right\} (a) \\ \psi_0 = 5: & \left\{ \begin{aligned} 1 - \beta &= 1.939 \times 10^{-4} \mathfrak{M}^2 \\ \mathfrak{R} &= 0.01380 (1 - \beta)^{-1/6} \end{aligned} \right\} (b) \\ \psi_0 = 2: & \left\{ \begin{aligned} 1 - \beta &= 1.025 \times 10^{-3} \mathfrak{M}^2 \\ \mathfrak{R} &= 0.03267 (1 - \beta)^{-1/6} \end{aligned} \right\} (c) \\ \psi_0 = 0: & \left\{ \begin{aligned} 1 - \beta &= 2.251 \times 10^{-3} \mathfrak{M}^2 \\ \mathfrak{R} &= 0.0921 (1 - \beta)^{-1/6} \end{aligned} \right\} (d) \\ \psi_0 = -\infty: & \left\{ \begin{aligned} 1 - \beta &= 3.093 \times 10^{-3} \mathfrak{M}^2 \\ \mathfrak{R} &= \text{Undetermined} \\ (\mathfrak{R}_0 &= 2.7362 \mathfrak{R}_1) \end{aligned} \right\} (e) \end{aligned} \right\} (1 - \beta \ll 1). \quad (35)$$

11. *The accurate solution of Milne's problem.*—Milne considered the problem of determining the locus on the $(\mathfrak{R}, 1 - \beta)$ diagram of the representative point of a perfect-gas

standard-model configuration of given mass contracting from $\mathfrak{R} = \infty$ to $\mathfrak{R} = \mathfrak{R}_{\min} \geq 0$, corresponding to "the maximum density of which matter is capable." He concluded that the locus would be the straight line (the Eddington line) $1 - \beta = 1 - \beta\mathfrak{M}$, where $(1 - \beta\mathfrak{M})$ is the value of the relative radiation pressure corresponding to $\mathfrak{M}$ through Eddington's

TABLE 3
POINTS DETERMINING THE $(\mathfrak{R}, 1 - \beta)$ CURVES FOR CONSTANT MASS

$\mathfrak{M} = \frac{M\mu_e^2}{\odot}$		$\psi_0 = \infty$ ($n = \frac{3}{2}$) ($\mathfrak{R} = \mathfrak{R}_1$)	$\psi_0 = 5$	$\psi_0 = 2$	$\psi_0 = 0$	$\psi_0 = -\infty$ (Eddington or $n = 3$) ($\mathfrak{R}_0$)
0.0	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ \infty \end{array}$	$\begin{array}{l} 0 \\ \infty \end{array}$	$\begin{array}{l} 0 \\ \infty \end{array}$	$\begin{array}{l} 0 \\ \infty \end{array}$	$\begin{array}{l} 0 \\ \infty \end{array}$
0.1	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0864 \end{array}$	$\begin{array}{l} 0.00001939 \\ 0.1236 \end{array}$	$\begin{array}{l} 0.00001025 \\ 0.2217 \end{array}$	$\begin{array}{l} 0.00002251 \\ 0.548 \end{array}$	$\begin{array}{l} 0.00003093 \\ (0.2363) \end{array}$
0.2	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0686 \end{array}$	$\begin{array}{l} 0.00000776 \\ 0.0981 \end{array}$	$\begin{array}{l} 0.00004109 \\ 0.1760 \end{array}$	$\begin{array}{l} 0.0000900 \\ 0.4352 \end{array}$	$\begin{array}{l} 0.0001237 \\ (0.1876) \end{array}$
0.3	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0599 \end{array}$	$\begin{array}{l} 0.00001745 \\ 0.0857 \end{array}$	$\begin{array}{l} 0.0000922 \\ 0.1537 \end{array}$	$\begin{array}{l} 0.0002026 \\ 0.3801 \end{array}$	$\begin{array}{l} 0.0002783 \\ (0.1639) \end{array}$
0.4	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0544 \end{array}$	$\begin{array}{l} 0.00003103 \\ 0.0779 \end{array}$	$\begin{array}{l} 0.0001640 \\ 0.1396 \end{array}$	$\begin{array}{l} 0.0003602 \\ 0.3454 \end{array}$	$\begin{array}{l} 0.0004948 \\ (0.1489) \end{array}$
0.5	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0505 \end{array}$	$\begin{array}{l} 0.0000485 \\ 0.0723 \end{array}$	$\begin{array}{l} 0.0002562 \\ 0.1296 \end{array}$	$\begin{array}{l} 0.000563 \\ 0.3206 \end{array}$	$\begin{array}{l} 0.000773 \\ (0.1382) \end{array}$
0.6	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0475 \end{array}$	$\begin{array}{l} 0.0000698 \\ 0.0680 \end{array}$	$\begin{array}{l} 0.0003689 \\ 0.1220 \end{array}$	$\begin{array}{l} 0.000810 \\ 0.3017 \end{array}$	$\begin{array}{l} 0.001113 \\ (0.1301) \end{array}$
0.7	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0452 \end{array}$	$\begin{array}{l} 0.0000950 \\ 0.0646 \end{array}$	$\begin{array}{l} 0.000502 \\ 0.1159 \end{array}$	$\begin{array}{l} 0.001103 \\ 0.2866 \end{array}$	$\begin{array}{l} 0.001515 \\ (0.1235) \end{array}$
0.8	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0432 \end{array}$	$\begin{array}{l} 0.0001241 \\ 0.0618 \end{array}$	$\begin{array}{l} 0.000656 \\ 0.1108 \end{array}$	$\begin{array}{l} 0.001441 \\ 0.2741 \end{array}$	$\begin{array}{l} 0.001979 \\ (0.1182) \end{array}$
0.9	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0415 \end{array}$	$\begin{array}{l} 0.0001571 \\ 0.0594 \end{array}$	$\begin{array}{l} 0.000830 \\ 0.1066 \end{array}$	$\begin{array}{l} 0.001823 \\ 0.2636 \end{array}$	$\begin{array}{l} 0.002505 \\ (0.1136) \end{array}$
1.0	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0401 \end{array}$	$\begin{array}{l} 0.0001939 \\ 0.0574 \end{array}$	$\begin{array}{l} 0.001025 \\ 0.1029 \end{array}$	$\begin{array}{l} 0.002251 \\ 0.2545 \end{array}$	$\begin{array}{l} 0.003093 \\ (0.1097) \end{array}$
1.1	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0388 \end{array}$	$\begin{array}{l} 0.0002346 \\ 0.0556 \end{array}$	$\begin{array}{l} 0.001240 \\ 0.0997 \end{array}$	$\begin{array}{l} 0.002724 \\ 0.2465 \end{array}$	$\begin{array}{l} 0.003742 \\ (0.1063) \end{array}$
1.2	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0377 \end{array}$	$\begin{array}{l} 0.0002792 \\ 0.0540 \end{array}$	$\begin{array}{l} 0.001476 \\ 0.0968 \end{array}$	$\begin{array}{l} 0.003241 \\ 0.2395 \end{array}$	$\begin{array}{l} 0.004453 \\ (0.1032) \end{array}$
1.3	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0367 \end{array}$	$\begin{array}{l} 0.0003277 \\ 0.0526 \end{array}$	$\begin{array}{l} 0.001732 \\ 0.0943 \end{array}$	$\begin{array}{l} 0.003804 \\ 0.2332 \end{array}$	$\begin{array}{l} 0.00523 \\ (0.1005) \end{array}$
1.4	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0358 \end{array}$	$\begin{array}{l} 0.0003801 \\ 0.0513 \end{array}$	$\begin{array}{l} 0.002008 \\ 0.0920 \end{array}$	$\begin{array}{l} 0.004412 \\ 0.2275 \end{array}$	$\begin{array}{l} 0.00606 \\ (0.0981) \end{array}$
1.5	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0350 \end{array}$	$\begin{array}{l} 0.0004363 \\ 0.0501 \end{array}$	$\begin{array}{l} 0.002306 \\ 0.0899 \end{array}$	$\begin{array}{l} 0.00506 \\ 0.2223 \end{array}$	$\begin{array}{l} 0.00696 \\ (0.0958) \end{array}$
1.6	$\left\{ \begin{array}{l} 1 - \beta \\ \mathfrak{R} \end{array} \right.$	$\begin{array}{l} 0 \\ 0.0343 \end{array}$	$\begin{array}{l} 0.0004964 \\ 0.04905 \end{array}$	$\begin{array}{l} 0.002623 \\ 0.0880 \end{array}$	$\begin{array}{l} 0.00576 \\ 0.2176 \end{array}$	$\begin{array}{l} 0.00792 \\ (0.09380) \end{array}$

quartic equation. The line extends parallel to the $\mathfrak{R}$ -axis from infinity to that value of $\mathfrak{R}$ at which complete degeneracy is just beginning to develop at the center; the end-point on the $\mathfrak{R}$ -axis would be that value of $\mathfrak{R}$ which corresponds to $\mathfrak{M}$ through the mass-radius relation for a polytrope of index $n = \frac{3}{2}$ (the second of eqs. [35a]) appropriate to the ultimate "completely collapsed" (completely degenerate) configuration. Milne proposed to determine the course of the $(\mathfrak{R}, 1 - \beta)$ curves within the domain of degeneracy between the above-mentioned end-points by considering composite configurations consisting of unrelativistic completely degenerate $n = \frac{3}{2}$ cores surrounded by (standard model) $n = 3$

perfect-gas envelopes. The curves were to be carried from the intersection of the Eddington line with the degeneracy-criterion locus (open circles on Figs. 3 and 4, *b*) to the end-points on the $\mathfrak{R}$ -axis by allowing the fractional radius $\mathfrak{R}_{\text{core}}/\mathfrak{R}$ of the $n = \frac{3}{2}$ core to vary parametrically from 0 to 1. This proposal was carried through by Fairclough.

Now that tables of the Fermi-Dirac functions $F_{1/2}$ and $F_{3/2}$ have been made available, it is possible to solve Milne's problem accurately, as is done in the present paper, by taking exact account of the gradual setting-in of partial degeneracy in going toward the center of the star. Our Table 3 and Figures 3 and 4 give the accurate solution of Milne's

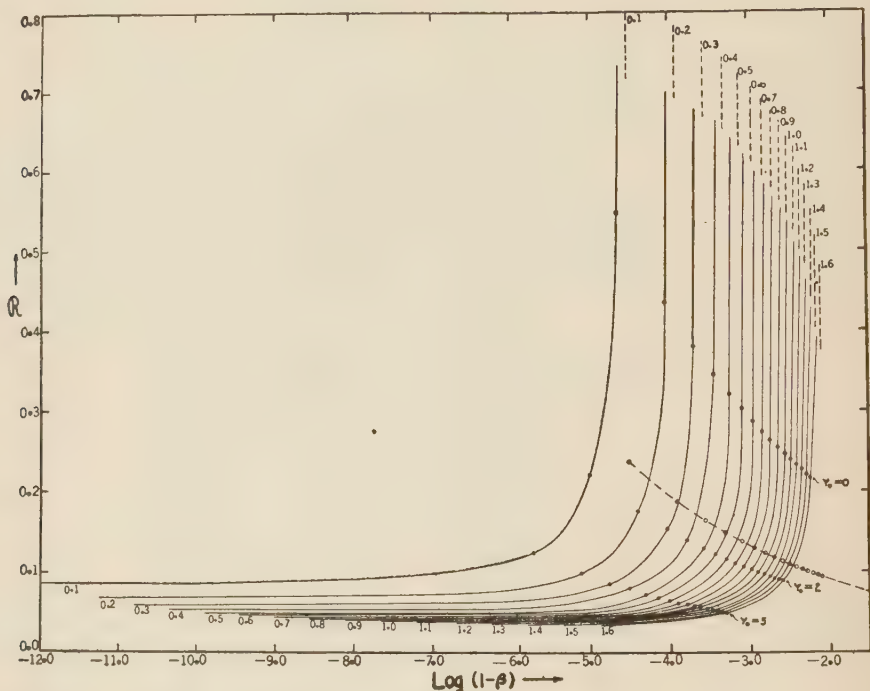


FIG. 3.—The Milne diagram. The accurate $(\mathfrak{R}, \log 1 - \beta)$ curves of constant mass in the domain of degeneracy. For each curve the value of the mass $\mathfrak{M} = M\mu_e^2/\odot$ is indicated beside the vertical asymptote (the Eddington line) and also below the horizontal asymptote (the line $\mathfrak{R} = R\mu_e/R\odot = \mathfrak{R}_1$). The foot of each Eddington line at the point $(\log 1 - \beta, \mathfrak{R}_0)$ is indicated by an open circle.

problem, restricted, however, to masses that are small enough that relativity effects may be neglected. For definiteness we may state that the error of the present theory for our values of $\psi_0 \geq 0$ does not exceed about 10 per cent for $\mathfrak{M} \equiv M\mu_e^2/\odot \leq 1.6$, or about 1 per cent for $\mathfrak{M} \leq \frac{1}{2}$. This estimate includes the error due to the neglect of $1 - \beta^2$, which decreases rapidly for $\mathfrak{M} < 1.6$ and/or $\psi_0 > 0$.

In Figure 3 the $(\mathfrak{R}, 1 - \beta)$ curves for the masses $\mathfrak{M} = 0.1, 0.2, \dots, 1.6$, for which the error should be less than 10 per cent, are plotted (from Table 3), using a logarithmic scale in $(1 - \beta)$ to avoid the crowding-together of the curves of small mass. The curves all have both vertical and horizontal asymptotes, the former being the Eddington line for mass $\mathfrak{M}$ and the latter the line $\mathfrak{R} = \mathfrak{R}_1$. The curved portion of each curve is determined by the three points corresponding to the numerical integrations for $\psi_0 = 0, 2$, and 5. Since there is some arbitrariness in the drawing of the curves, these three points, as well

as the asymptotes, are clearly indicated. The point $(\log_{10} [1 - \beta], \mathcal{R}_0)$ at the "foot" of each Eddington line is indicated as an open circle. The segment of a curve drawn through the latter points is very nearly (≤ 10 per cent for the present relatively small masses) a portion of the $(\mathcal{R}_0, 1 - \beta)$ curve¹⁰ defining the domain of degeneracy on the Milne diagram in the theory of composite configurations having degenerate cores. In Figure 4, *a*, the curves of Figure 3 are transformed by the use of a natural, instead of a logarithmic, scale for $(1 - \beta)$; the figure is, therefore, directly comparable to Chandrasekhar's figure. In Figure 4, *b*, the scale of abscissas of Figure 4, *a*, is magnified to show some of the

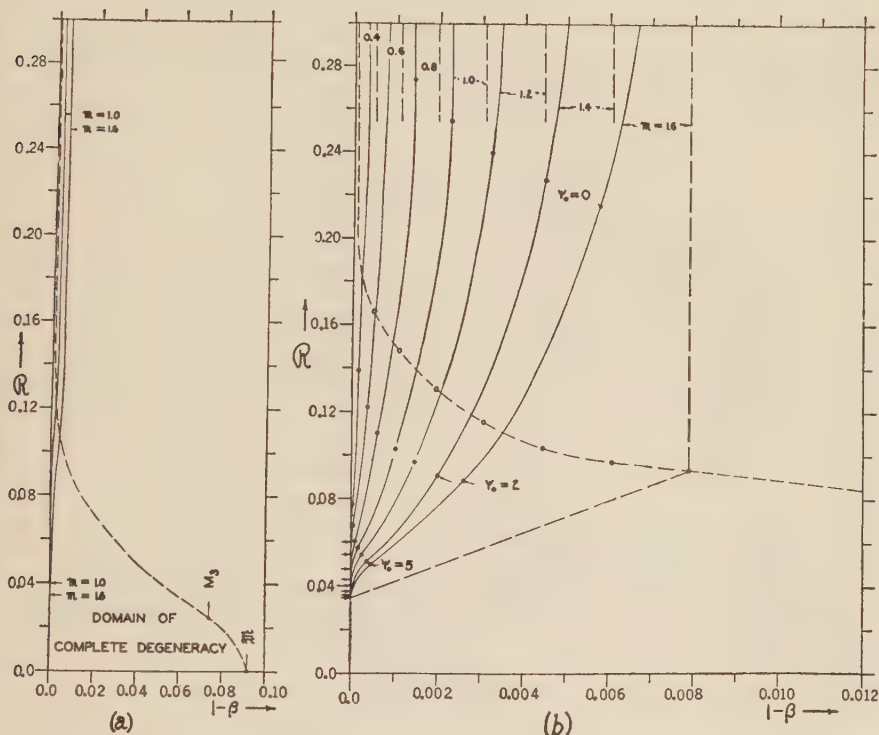


FIG. 4.—The Milne diagram (natural scale). The dashed curve is Chandrasekhar's $(\mathcal{R}_0, 1 - \beta)$ curve defining the domain of degeneracy. The arrows on the $\mathcal{R}$ -axis indicate, from top to bottom, the end-points $(0, \mathcal{R}_1)$ for $M = 0.4, 0.6, 0.8, \dots, 1.6$, respectively in (b).

$(\mathcal{R}, 1 - \beta)$ curves more clearly. On the natural scale for $(1 - \beta)$ these curves approach the end-points $(0, \mathcal{R}_1)$ on the $\mathcal{R}$ -axis vertically instead of horizontally as on the logarithmic scale.

On Milne's theory the $(\mathcal{R}, 1 - \beta)$ curve of constant mass on Figure 4, *b*, for $M = 1.6$, for example, would be the Eddington line labeled " $M = 1.6$ " down to the open circle corresponding to $\mathcal{R}_0$, then some curve ending at $\mathcal{R}_1$ on the $\mathcal{R}$ -axis, the latter curve being represented as a dashed straight line in the figure. The error in $\mathcal{R}$, as read from the latter straight line instead of from the accurate curve, is of the order of from -100 per cent to -200 per cent; and the Milne curves would be expected to lie below, rather than above, the straight line, at least near the junction with the Eddington line.

¹⁰ S. Chandrasekhar, *M.N.*, 95, 226, 1935, Fig. 3, or *Stellar Structure*, Fig. 34.

II. THE PARTIALLY DEGENERATE ISOTHERMAL GAS SPHERE

12. *The fundamental differential equation.*—By definition of the isothermal gas sphere, $T = \text{constant}$. Hence the equation of state of the model is simply the Fermi-Dirac equation of state (4), with the radiation pressure $p_r = (a/3)T^4$ added to the gas pressure p_g to give the total pressure:

$$\left. \begin{aligned} \rho &= B_1 F_{1/2}(\psi) \equiv |B_1| \mu_e T^{3/2} F_{1/2}(\psi), \\ P &= A_1 \mathfrak{F}_{3/2}(\psi) + \frac{a}{3} T^4 \equiv |A_1| T^{5/2} \mathfrak{F}_{3/2}(\psi) + \frac{a}{3} T^4, \end{aligned} \right\} \quad (36)$$

where A_1 , B_1 , $|A_1|$, and $|B_1|$ are constants defined by equations (5) and (6). Using the transformations

$$r = a_1 \xi, \quad (37)$$

where

$$a_1 = \left[\frac{A_1}{4\pi G B_1^2} \right]^{1/2} \equiv \frac{|a_1|}{\mu_e T^{1/4}} = \frac{1.495}{\mu_e T^{1/4}} R_\odot, \quad (38)$$

we obtain, as our fundamental differential equation,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\psi}{d\xi} \right) = -F_{1/2}(\psi), \quad \text{or} \quad \frac{d^2\psi}{d\xi^2} + \frac{2}{\xi} \frac{d\psi}{d\xi} = -F_{1/2}(\psi), \quad (39)$$

subject to the usual boundary conditions (cf. eqs. [19]). In the present case, according to the first of equations (36),

$$\frac{\rho_c}{\mu_e T^{3/2}} = |B_1| F_{1/2}(\psi_0), \quad \text{or} \quad \psi_0 \text{ specifies } \frac{\rho_c}{\mu_e T^{3/2}}, \quad (40)$$

which is analogous to the last of specifications (20) for the standard model. As in the latter case, there is no homology relation.

13. *The density at any point in terms of the central density.*—By equation (19), the first of equations (36), and the assumption that the parameters T and μ_e are constant throughout the configuration we find

$$\frac{\rho}{\rho_c} = \frac{F_{1/2}(\psi)}{F_{1/2}(\psi_0)}. \quad (41)$$

14. *The mass relation.*—Proceeding as in § 4, we find

$$M(\xi) = -C_1 \xi^2 \frac{d\psi}{d\xi} \equiv -|C_1| \mu_e^{-2} T^{3/4} \xi^2 \frac{d\psi}{d\xi}, \quad (42)$$

where

$$|C_1| = 4\pi |a_1|^3 |B_1| = 6.47 \times 10^{-8} \odot. \quad (43)$$

15. *The relation between the mean and the central density.*—Proceeding as in § 5, we find from equations (36), (37), and (42) the expression

$$\frac{\rho_c}{\bar{\rho}} = -F_{1/2}(\psi_0) \frac{\xi_1}{3\psi'(\xi_1)}. \quad (44)$$

16. *The limiting cases of very low and of very high central degeneracy.*—In the limit of very low central degeneracy our present partially degenerate isothermal configurations reduce to the classical isothermal gas sphere (polytrope with $n = \infty$), and in the limit of very high central degeneracy they reduce to precisely the same limit as do the partially

degenerate standard-model configurations, namely, to the low-density limit (polytrope with $n = \frac{3}{2}$) of the white-dwarf configurations.

17. *The numerical integrations.*—Seven integrations of the partially degenerate isothermal equation (39) for the values $\psi_0 = 0, 2, 3, 5, 10, 20$, and 100 have been carried through with from seven to nine significant figures in the integration variable $\varphi \equiv \psi\xi$, each integration being carried at least as far as the point where $\rho/\rho_c < 10^{-4}$. Tables of measures of the same three physical quantities as in Table 1 for the standard model are given elsewhere.¹¹ These tables are summarized by Table 4 and Figure 5. In Figure 5 the density-distribution-curves for partially degenerate isothermal configurations having

TABLE 4
THE PARTIALLY DEGENERATE ISOTHERMAL GAS SPHERE WITH
CENTRAL DENSITY $\rho_c = 10^3$ GM/CM³

ψ_0	$T_{\mu_e}^{2/3} \times 10^6$ (° K)	$\rho/\rho_c = 10^{-1}$	$\rho/\rho_c = 10^{-2}$		$\rho/\rho_c = 10^{-3}$		$\rho/\rho_c = 10^{-4}$	
		$r_{\mu_e}^{5/6} M(r_{\mu_e}^{5/6})^{5/2}$	Multiple of "•" Value	$r_{\mu_e}^{5/6} M(r_{\mu_e}^{5/6})^{5/2}$	Multiple of "•" Value	$r_{\mu_e}^{5/6} M(r_{\mu_e}^{5/6})^{5/2}$	Multiple of "•" Value	$r_{\mu_e}^{5/6} M(r_{\mu_e}^{5/6})^{5/2}$
$-\infty (n = +\infty)$	∞	$\begin{Bmatrix} \infty \\ \infty \end{Bmatrix}$	$\begin{pmatrix} 2.5400 \\ 2.6179 \end{pmatrix}$	$\begin{Bmatrix} \infty \\ \infty \end{Bmatrix}$	$\begin{pmatrix} 7.2929 \\ 6.1469 \end{pmatrix}$	$\begin{Bmatrix} \infty \\ \infty \end{Bmatrix}$	$\begin{pmatrix} 24.910 \\ 19.780 \end{pmatrix}$	$\begin{Bmatrix} \infty \\ \infty \end{Bmatrix}$
0	29.7	$\begin{Bmatrix} 0.14131 \\ 0.46934 \end{Bmatrix}$	$\begin{pmatrix} 2.4159 \\ 2.3397 \end{pmatrix}$	$\begin{pmatrix} 0.34139 \\ 1.0980 \end{pmatrix}$	$\begin{pmatrix} 6.8969 \\ 5.1680 \end{pmatrix}$	$\begin{pmatrix} 0.9746 \\ 2.4255 \end{pmatrix}$	$\begin{pmatrix} 24.339 \\ 17.188 \end{pmatrix}$	$\begin{pmatrix} 3.4393 \\ 8.0670 \end{pmatrix}$
2	12.4	$\begin{Bmatrix} 0.09791 \\ 0.17087 \end{Bmatrix}$	$\begin{pmatrix} 2.1426 \\ 1.8729 \end{pmatrix}$	$\begin{pmatrix} 0.20979 \\ 0.32002 \end{pmatrix}$	$\begin{pmatrix} 5.7986 \\ 3.3952 \end{pmatrix}$	$\begin{pmatrix} 0.5678 \\ 0.5801 \end{pmatrix}$	$\begin{pmatrix} 22.554 \\ 11.781 \end{pmatrix}$	$\begin{pmatrix} 2.2084 \\ 2.0129 \end{pmatrix}$
3	9.13	$\begin{Bmatrix} 0.08825 \\ 0.13136 \end{Bmatrix}$	$\begin{pmatrix} 1.9818 \\ 1.6658 \end{pmatrix}$	$\begin{pmatrix} 0.17489 \\ 0.21882 \end{pmatrix}$	$\begin{pmatrix} 4.9846 \\ 2.5957 \end{pmatrix}$	$\begin{pmatrix} 0.4399 \\ 0.3410 \end{pmatrix}$	$\begin{pmatrix} 20.767 \\ 8.5606 \end{pmatrix}$	$\begin{pmatrix} 1.8328 \\ 1.1245 \end{pmatrix}$
5	5.81	$\begin{Bmatrix} 0.07867 \\ 0.10035 \end{Bmatrix}$	$\begin{pmatrix} 1.7243 \\ 1.4122 \end{pmatrix}$	$\begin{pmatrix} 0.13565 \\ 0.14171 \end{pmatrix}$	$\begin{pmatrix} 3.5131 \\ 1.7282 \end{pmatrix}$	$\begin{pmatrix} 0.2764 \\ 0.1734 \end{pmatrix}$	$\begin{pmatrix} 14.2032 \\ 3.3590 \end{pmatrix}$	$\begin{pmatrix} 1.1174 \\ 0.3371 \end{pmatrix}$
10	2.98	$\begin{Bmatrix} 0.07200 \\ 0.08347 \end{Bmatrix}$	$\begin{pmatrix} 1.4332 \\ 1.2107 \end{pmatrix}$	$\begin{pmatrix} 0.10319 \\ 0.10105 \end{pmatrix}$	$\begin{pmatrix} 2.0627 \\ 1.2672 \end{pmatrix}$	$\begin{pmatrix} 0.14852 \\ 0.10576 \end{pmatrix}$	$\begin{pmatrix} 3.5171 \\ 1.2963 \end{pmatrix}$	$\begin{pmatrix} 0.2532 \\ 0.1082 \end{pmatrix}$
20	1.50	$\begin{Bmatrix} 0.07002 \\ 0.07926 \end{Bmatrix}$	$\begin{pmatrix} 1.2916 \\ 1.1397 \end{pmatrix}$	$\begin{pmatrix} 0.09044 \\ 0.09033 \end{pmatrix}$	$\begin{pmatrix} 1.5412 \\ 1.1562 \end{pmatrix}$	$\begin{pmatrix} 0.10792 \\ 0.09163 \end{pmatrix}$	$\begin{pmatrix} 1.8733 \\ 1.1592 \end{pmatrix}$	$\begin{pmatrix} 0.13117 \\ 0.09188 \end{pmatrix}$
100	0.300	$\begin{Bmatrix} 0.06949 \\ 0.07815 \end{Bmatrix}$	$\begin{pmatrix} 1.2265 \\ 1.1142 \end{pmatrix}$	$\begin{pmatrix} 0.08522 \\ 0.08707 \end{pmatrix}$	$\begin{pmatrix} 1.3003 \\ 1.1190 \end{pmatrix}$	$\begin{pmatrix} 0.09035 \\ 0.08745 \end{pmatrix}$	$\begin{pmatrix} 1.3462 \\ 1.1193 \end{pmatrix}$	$\begin{pmatrix} 0.09354 \\ 0.08747 \end{pmatrix}$
$+\infty (n = \frac{3}{2})$	0	$\begin{Bmatrix} 0.06947 \\ 0.07811 \end{Bmatrix}$	$\begin{pmatrix} 1.2242 \\ 1.1134 \end{pmatrix}$	$\begin{pmatrix} 0.08504 \\ 0.08696 \end{pmatrix}$	$\begin{pmatrix} 1.2835 \\ 1.1175 \end{pmatrix}$	$\begin{pmatrix} 0.08916 \\ 0.08728 \end{pmatrix}$	$\begin{pmatrix} 1.2970 \\ 1.1176 \end{pmatrix}$	$\begin{pmatrix} 0.09010 \\ 0.08729 \end{pmatrix}$

$\rho_c = 10^3$ gm/cm³ are plotted against radial distance $r_{\mu_e}^{5/6}$ in both centimeter and solar units. The curve for the limiting case, $\psi_0 \rightarrow -\infty$, is the sensibly straight line $\rho/\rho_c = 1$ on the scale of Figure 5. For the other limiting case, $\psi_0 \rightarrow +\infty$, the curve cuts the $r_{\mu_e}^{5/6}$ axis abruptly at 6.28×10^9 cm = $0.0904 R_\odot$ with the slope $\theta'_{3/2}(\eta_1) = -0.2033013$. For finite positive values of ψ_0 the solutions have no mathematically finite boundaries, but they approach the finite boundary of, as well as the curve of, the latter limiting case asymptotically as $\psi_0 \rightarrow +\infty$, so that it is possible to define a physical boundary for only moderately large values of ψ_0 . Note that the curve for $\psi_0 = 100$ is indistinguishable in Figure 5 from that for $\psi_0 = +\infty$.

After a certain low density has been reached, the solutions can be continued analytically by means of the formula

$$\psi(\xi) = a + b\xi^{-1} - c^2 \log \xi \quad (45)$$

¹¹ See n. 1, above.

where a , b , and c^2 are arbitrary constants of integration, the latter being inherently positive.

In Table 4 are given the values of the radius $r\mu_e^{5/6}$, of the mass $M(r)\mu_e^{5/2}$, and of the isothermal temperature $T\mu_e^{2/3}$ for the various solutions ψ_0 , all calculated for central density $\rho_c = 10^3 \text{ gm/cm}^3$.³ The radii and masses are each given both in solar units and (in parentheses) in terms of their respective values at $r = r^*$, defined as where $\rho/\rho_c = 10^{-1}$. Pairs of values of $r\mu_e^{5/6}$ and $M(r)\mu_e^{5/2}$ are given for each of the four values: $\rho/\rho_c = 10^{-1}$, 10^{-2} , 10^{-3} , and 10^{-4} , the value of $r\mu_e^{5/6}$ being given above, and that of $M(r)\mu_e^{5/2}$ below, for each solution ψ_0 . Between successive columns of pairs of radius and mass values are given, in parentheses, the multiple that each quantity $r\mu_e^{5/6}$ and $M(r)\mu_e^{5/2}$ is of its value at $r^*\mu_e^{5/6}$

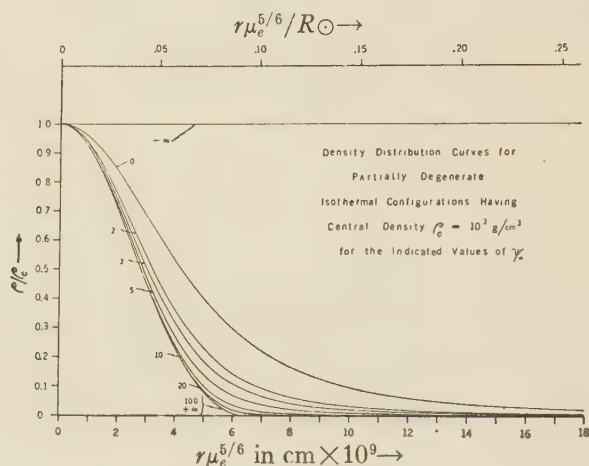


FIG. 5.—Relative density distributions for partially degenerate isothermal configurations

Table 4 may be converted immediately for use for any other value of ρ_c or of T by noting equations (37), (38), (40), and (42). Specifically, Table 4 may be transformed by the following relations which hold for a given solution ψ_0 :

$$r\mu_e^{5/6} \propto \rho_c^{-1/6}; \quad M(r)\mu_e^{5/2} \propto \rho_c^{1/2}; \quad T\mu_e^{2/3} \propto \rho_c^{2/3}. \quad (46)$$

III. APPLICATIONS

18. *The relevant range of M , R , and μ_e values.*—For applications of the theory of the partially degenerate standard model we see from Figure 3 that we should look for stars having

$$\frac{1}{30}R_\odot \leq R \leq 0.4R_\odot, \quad M \leq 1.6\odot \quad (\mu_e = 1). \quad (47)$$

The theory is perfectly applicable, of course, to stars having $R > 1/30R_\odot$ and $M < 1.6\odot$ but whose representative points are far removed from the curved portion of Figure 3; but, since either the classical standard model $n = 3$ or the $n = \frac{3}{2}$ model is applicable to such stars with sufficient accuracy, we do not consider them here. We note that the effect of $\mu_e > 1$ is to multiply $\mathcal{R}$ by μ_e and $\mathcal{M}$ by μ_e^2 , so that the representative point on Figure 3 moves upward and to the right in the direction of nondegeneracy. For $\mu_e = 2$ the specification (47) becomes

$$\frac{1}{30}R_\odot \leq R \leq 0.2R_\odot, \quad M \leq 0.4\odot \quad (\mu_e = 2). \quad (48)$$

19. *The subdwarf stars.*—Among the subdwarf stars, Kuiper¹² has recently discovered¹³ two extreme cases, Wolf 134 and Wolf 1037, of spectral type K3 and K5, respectively. The absolute magnitudes are roughly +12.6 and +12.5, which means that they are about 5.7 and 4.5 magnitudes, respectively, below the main sequence. If they are of the same color as main-sequence K stars, their radii should be about one-tenth as great, or about $0.08 R_{\odot}$; if their color is not the same, it would be expected to be slightly bluer, corresponding to slightly smaller radii. There is no direct way of determining the masses, but theoretical considerations lead to the expectation of low hydrogen content ($\mu_e \approx 2$) and masses of the order of $0.1 \odot$. We shall consider the following three sets of numerical values:

$$\left. \begin{array}{ll} (a) & R = 0.05 R_{\odot}, \quad M = 0.05 \odot ; \\ & \mathfrak{R} = 0.10, \quad \mathfrak{M} = 0.2 ; \\ (b) & R = 0.08 R_{\odot}, \quad M = 0.1 \odot ; \\ & \mathfrak{R} = 0.16, \quad \mathfrak{M} = 0.4 ; \\ (c) & R = 0.16 R_{\odot}, \quad M = 0.3 \odot ; \\ & \mathfrak{R} = 0.32, \quad \mathfrak{M} = 1.2 ; \end{array} \right\} \quad (49)$$

corresponding to (a) improbably high central degeneracy, (b) the most probable values, and (c) improbably low central degeneracy,¹⁴ all for $\mu_e = 2$. The values of $\log (1 - \beta)$ and the approximate values of ψ_0 are read from Figure 3 and used to calculate $\log \rho_c$ and $\log T_c$ on the basis of the classical Eddington standard model¹⁵ and on the basis of the present partially degenerate standard model as follows:

$$\left. \begin{array}{ll} (a) & \psi_0 = +5.0 \\ & \log (1 - \beta) = -5.20 \\ (b) & \psi_0 = +1.7 \\ & \log (1 - \beta) = -3.70 \\ (c) & \psi_0 = -0.5 \\ & \log (1 - \beta) = -2.46 \end{array} \right\} \quad (50)$$

Classical (Eddington) Standard Model

$$\left. \begin{array}{lll} (a) & \log \rho_c = 4.49 & (b) \quad \log \rho_c = 4.17 \quad (c) \quad \log \rho_c = 3.75 \\ & \log T_c = 7.60 & \log T_c = 7.69 \quad \log T_c = 7.87 \end{array} \right\} \quad (51)$$

Partially Degenerate Standard Model

$$\left. \begin{array}{lll} (a) & \log \rho_c = 3.7 & (b) \quad \log \rho_c = 3.8 \quad (c) \quad \log \rho_c = 3.6 \\ & \log T_c = 7.0 & \log T_c = 7.5 \quad \log T_c = 7.8 \end{array} \right\} \quad (52)^{16}$$

¹² I am indebted to Dr. Kuiper for kindly communicating to me his unpublished estimates of the best values (as of April, 1941) of the spectra, masses, and radii of the subdwarfs and also of the old novae discussed in § 20. I am also indebted to him for helpful discussions and suggestions regarding subdwarf and white-dwarf stars and for access to his unpublished paper on this subject presented before the 1939 Paris meeting of the I.A.U.

¹³ G. P. Kuiper, *Ap. J.*, **91**, 269, 1940.

¹⁴ High central degeneracy is not directly correlated with high mean density, for two reasons: (1) the degeneracy-criterion locus is not vertical, and (2) $\rho_c/\bar{\rho}$ decreases with increasing degeneracy, according to Table 2. Thus, $\mathfrak{R} = 0.10$, $\mathfrak{M} = 1.2$ is more dense than (a), but $\psi_0 = +2.0$, as compared with +5.0; also $\mathfrak{R} = 0.32$, $\mathfrak{M} = 0.20$ is less dense than (c), but $\psi_0 = +0.5$, as compared with -0.5.

¹⁵ Using the formulae: $\rho_c = 54.18 \bar{\rho}$ and $T_c = 19.72 \times 10^6 \beta \mathfrak{M}/\mathfrak{R}$.

¹⁶ The values of $\log \rho_c$ are uncertain by about ± 0.1 in (a) and (b) and by a somewhat larger amount in (c), on account of the uncertainty in the interpolation of ψ_0 . The uncertainty in $\log T_c$ is about one-third that in $\log \rho_c$.

We note that all three cases, with the possible exception of case (c), fall in the curved region of Figure 3, which is just the region in which the present theory is required. Comparing equations (51) and (52) with Figure 1, we see that the representative point for the center of these subdwarfs falls near the $Q = -10$ locus, the degeneracy-criterion locus, and the $q = -10$ locus, respectively, in cases (a), (b), and (c). The classical theory predicts the central density too high by about 0.8, 0.3, and 0.1 in $\log \rho_c$ in cases (a), (b), and (c), respectively. The error in $\log T_c$ is about two-thirds as great and is in the same sense.

There are several other subdwarf stars in Kuiper's lists¹⁷ that are less extreme than the two discussed above but which probably fall in the realm of the present theory.

20. *The old novae.*—Humason's investigation¹⁸ of sixteen old novae gives the spectrum as type O and the mean absolute visual magnitude, $\bar{M}_v$, as about +3.0. From theoretical considerations Kuiper¹² expects these stars to have low hydrogen content ($\mu_e \sim 2$). Accordingly, he considers $M = 0.3 - 0.5 \odot$ as more probable for $\bar{M}_v = +3$ than the value $M = \odot$ adopted by Humason. He finds $\log R = -0.43, -0.71$, or -0.91 , corresponding, respectively, to temperature $25,000^\circ$, $50,000^\circ$, or $100,000^\circ$, the first and last values being limits and the second the most probable. Only if the radii and masses should both be near their lower limits would the representative points fall in the curved portion of Figure 3, if $\mu_e = 2$. In no case would the representative point, $\log \rho_c, \log T_c$, on Figure 1 fall below the degeneracy-criterion locus. In the most extreme case we have for the partially degenerate and for the Eddington models:

$$\left. \begin{aligned} R &= 0.12R_\odot, & M &= 0.3\odot; \\ \mathfrak{R} &= 0.24, & \mathfrak{M} &= 1.2; \\ \log(1-\beta) &= -2.48; \\ \psi_0 &= 0.00; \\ &(\text{P.D.}) (\text{ED.}) \\ \log \rho_c &= 3.97, 4.12; \\ \log T_c &= 7.92, 7.99 & (\mu_e = 2), \end{aligned} \right\} \quad (53)$$

corresponding to a point about 0.2 units above the degeneracy-criterion locus. In the most probable case we have:

$$\left. \begin{aligned} R &= 0.20R_\odot, & M &= 0.4\odot; \\ \mathfrak{R} &= 0.40, & \mathfrak{M} &= 1.6; \\ \log(1-\beta) &= -2.12; \\ \psi_0 &\approx -1; \\ &(\text{P.D.}) (\text{ED.}) \\ \log \rho_c &= 3.6, 3.58; \\ \log T_c &= 7.9, 7.90 & (\mu_e = 2), \end{aligned} \right\} \quad (54)$$

corresponding to a point very nearly on the $q = -10$ locus of Figure 1. Thus, the Eddington theory should be adequate in the most probable case. We conclude that the partially degenerate standard model would be required for the old novae only under the following improbable circumstances: (a) both radii and masses fall at the lower limits; (b) the radii and masses lie anywhere between the above limits, and $\mu_e = 1$; or (c) the radii and masses have their most probable values, and $\mu_e < 1.6$.

21. *The validity of the classical standard model for a star of given mass and radius.*—We note the convenient coincidence that for $\mu_e = 2$ the Eddington model can be used to test

¹⁷ G. P. Kuiper, *Ap. J.*, **91**, 269, 1940; **92**, 128, 1940, Table 1.

¹⁸ M. L. Humason, *Ap. J.*, **88**, 228, 1938.

accurately its own validity. In cases in which we suspect that the perfect-gas theory may fail, we expect $\mu_e \sim 2$, in which case the representative point ($\log \rho_c, \log T_c$) calculated on the classical model, is found empirically to bear very nearly the same relation to the degeneracy-criterion locus as when calculated on the partially degenerate standard model, even though the actual values of $\log \rho_c$ and $\log T_c$ may be greatly overestimated by the classical theory.

If $\mu_e = 1$ instead of $\mu_e = 2$, then the representative point is actually shifted horizontally to the left by 0.3 units on Figure 1 in the direction of nondegeneracy, but the Eddington theory predicts that it is moved by the same amount vertically downward in the direction of degeneracy. Thus, regardless of the value of μ_e (which must lie in the range 1.00 to 2.03 for a Russell mixture), we can say that, if the Eddington values of $\log \rho_c$ and $\log T_c$ indicate a point safely above the degeneracy-criterion locus, the Eddington model may be used with confidence (at least for $\mathfrak{M} \leq 1.6$). This will be the case if

$$\left. \begin{array}{ll} \log T_c > \frac{2}{3} \log \rho_c + 5.560, & \text{or} \quad \log T_c \gg \frac{2}{3} \log \rho_c + 5.080. \end{array} \right\} \quad (55)$$

(above the $q = -10$ locus) (far above degeneracy-criterion locus)

Otherwise the representative point should be calculated as in equations (49), (50), and (52); if $\log \rho_c$ and $\log T_c$ are not required, the central degeneracy (ψ_0) can be estimated by mere inspection of Figure 3 more exactly than by the use of the Eddington theory.

The classical standard model is probably adequate for main-sequence M-dwarf stars such as Krüger 60A and Krüger 60B. The mass of the latter is about the same as that of the subdwarfs, for which we found the partially degenerate standard model to be required; but the radius is so much larger (of the order of $0.4R_\odot$ instead of $0.1R_\odot$) that even for $\mu_e = 1$ the representative point on Figure 3 falls somewhat above the sharply curved portions of the ($\mathfrak{R}, 1 - \beta$) curves at $\psi_0 \sim 0$. If $\mu_e > 1$, as seems likely, then the central degeneracy is much less and should be negligible.

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